

# VECTOR BUNDLES AND FINITE COVERS

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ABSTRACT. We prove that, up to twist, every vector bundle on a smooth projective curve arises from the direct image of the structure sheaf of a smooth, connected branched cover.

## 1. INTRODUCTION

Associated to a finite flat morphism  $\phi : X \rightarrow Y$  is the vector bundle  $\phi_* \mathcal{O}_X$  on  $Y$ . This is the bundle whose fiber over  $y \in Y$  is the vector space of functions on  $\phi^{-1}(y)$ . It is a natural problem to understand which vector bundles on  $Y$  arise in this way. An equivalent formulation of the problem is to understand which vector bundles on  $Y$  admit the structure of an  $\mathcal{O}_Y$ -algebra.

Pull-back of functions gives a natural map  $\mathcal{O}_Y \rightarrow \phi_* \mathcal{O}_X$ . In characteristic not dividing the degree  $d$  of the map, it admits a left-inverse  $\phi_* \mathcal{O}_X \rightarrow \mathcal{O}_Y$  given by  $1/d$  times the trace map. As a result, the bundle  $\phi_* \mathcal{O}_X$  contains  $\mathcal{O}_Y$  as a direct summand. The dual of the remaining direct summand is called the *Tschirnhausen bundle* and denoted by  $E = E_\phi$  (The dual is taken as a convention.) We are thus reduced to understanding which vector bundles arise as Tschirnhausen bundles.

If we allow  $X$  to be arbitrarily singular, then every rank  $(d - 1)$  vector bundle on  $Y$  arises as a Tschirnhausen bundle. Indeed, given a vector bundle  $E$  on  $Y$ , we may take  $X$  to be the non-reduced scheme which is the first order neighborhood of the zero section in the total space of  $E$ .

When we require  $X$  to be reasonable, for instance smooth and connected, the classification of Tschirnhausen bundles becomes challenging. A few restrictions on  $E$  are immediate. Since  $X$  is connected, we must have  $H^0(Y, E^\vee) = 0$ . Furthermore, since the branch divisor is the vanishing locus of a section of  $(\det E)^{\otimes 2}$ , we must have  $H^0(Y, (\det E)^{\otimes 2}) \neq 0$ . This reflects the positivity properties of Tschirnhausen bundles, true in all dimensions, as shown in [11, 15, 16]. The precise necessary and sufficient conditions for being Tschirnhausen are unknown, and quite delicate even when  $Y$  is a curve.

Let us take the simplest non-trivial example:  $Y = \mathbf{P}^1$ . For  $d = 2$ , the necessary and sufficient condition for  $E = \mathcal{O}(n)$  to be a Tschirnhausen bundle is that  $n \geq 1$ . For  $d = 3$ , the story is more interesting, and originates from work of Maroni in [12]. In this case, necessary and sufficient conditions for  $E = \mathcal{O}(m) \oplus \mathcal{O}(n)$  with  $m \leq n$  to be a Tschirnhausen bundle are  $0 < n \leq 2m$ . Characterizing Tschirnhausen bundles for smooth, connected covers of  $\mathbf{P}^1$  quickly becomes difficult as  $d$  increases.

The problem of characterizing Tschirnhausen bundles on  $\mathbf{P}^1$  has attracted the attention of several mathematicians; see for example [1, 5, 13, 17]. Historically, it has been called the problem of classifying the *scrollar invariants* of finite covers. Writing  $E_\phi = \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_{d-1})$ , the scrollar invariants are the integers  $a_1, \dots, a_{d-1}$ .

A part of the picture for  $Y = \mathbf{P}^1$  emerging from the collective work of several authors indicates that if  $E$  is too “unbalanced” (e.g., if the largest summand’s degree is too close to the total

degree) then it cannot be a Tschirnhausen bundle for a smooth, connected cover. In light of this understanding, it is reasonable to ask whether every vector bundle arises, provided we allow the freedom of twisting by line bundles.

**Question 1.1.** *Let  $Y$  be a smooth projective variety and  $E$  a vector bundle on  $Y$ . Is  $E \otimes L$  the Tschirnhausen bundle of a smooth, connected branched cover  $\phi : X \rightarrow Y$  for some line bundle  $L$  on  $Y$ ?*

To our knowledge, the work of Ballico [1] comes closest to answering Question 1.1 for  $Y = \mathbf{P}^1$ . He shows that the question has an affirmative answer for “half” of the bundle  $E$ —when  $d$  is even, the smallest  $d/2$  scroller invariants can be arbitrarily specified up to a simultaneous twist.

We answer Question 1.1 completely for all smooth projective curves  $Y$ . Our main result is the following stronger statement.

**Theorem 1.2 (Main).** *Let  $Y$  be a smooth projective curve and let  $E$  be a vector bundle on  $Y$ . There exists an integer  $n$  (depending on  $E$ ) such that for any line bundle  $L$  on  $Y$  of degree at least  $n$ , the bundle  $E \otimes L$  is the Tschirnhausen bundle of a finite cover  $\phi : X \rightarrow Y$  with  $X$  smooth.*

For affine curves, we get the following corollary.

**Corollary 1.3.** *Suppose  $Y$  is a smooth affine curve, and  $E$  is a vector bundle on  $Y$ . Then  $E$  is the Tschirnhausen bundle for some map  $\phi : X \rightarrow Y$ , with  $X$  smooth and connected.*

*Proof.* Extend  $E$  to a vector bundle  $E'$  on the smooth projective compactification  $Y'$  of  $Y$ . Apply Theorem 1.2 to  $E'$ , twisting by a sufficiently positive line bundle  $L$  on  $Y'$  whose divisor class is supported on the complement  $Y' \setminus Y$ . We obtain a smooth curve  $X'$  and a map  $\phi : X' \rightarrow Y'$  whose Tschirnhausen bundle is  $E' \otimes L$ ; letting  $X = \phi^{-1}(Y)$ , we obtain the corollary.  $\square$

The method of proof of Theorem 1.2 yields a basic result relating the moduli branched covers of  $Y$  and the moduli of vector bundles on  $Y$ . Let  $H_{d,g}(Y)$  denote the Hurwitz space of degree  $d$  and genus  $g$  branched covers of  $Y$  and  $M_{d-1,d+g-1}(Y)$  the moduli space of semi-stable vector bundles of rank  $d - 1$  and degree  $d + g - 1$  on  $Y$ .

**Theorem 1.4.** *Suppose  $g_Y \geq 2$ . If  $g$  is sufficiently large (depending on  $Y$  and  $d$ ), then the Tschirnhausen bundle of a general degree  $d$  and genus  $g$  branched cover of  $Y$  is stable. Moreover, the rational map  $H_{d,g}(Y) \dashrightarrow M_{d-1,d+g-1}(Y)$  defined by  $\phi \mapsto E_\phi$  is dominant.*

*The same statement holds for  $g_Y = 1$ , with “stable” replaced with “regular poly-stable.”*

Theorem 1.4 is Corollary 3.10 in the main text.

Theorem 1.4 was proved for  $d \leq 5$  by Kanev [8, 9, 10], using the structure theorems of finite covers of low degree [3, 4]. The validity of Theorem 1.4 for low  $g$  is an interesting open problem. In particular, it would be nice to know whether  $\phi \rightarrow E_\phi$  is dominant as soon as we have  $\dim H_{d,g}(Y) \geq \dim M_{d-1,d+g-1}$ .

Our interest in Question 1.1 for curves originated partly in the study of cycles on  $H_{d,g}(Y)$ . For a vector bundle  $E$  on  $Y$ , define the *Maroni locus*  $M(E) \subset H_{d,g}(Y)$  as the locally closed subset that parametrizes covers with Tschirnhausen bundle isomorphic to  $E$ . This notion generalizes the Maroni loci for  $Y = \mathbf{P}^1$  studied in [6] and [14]. As a consequence of the method of proof of the main theorem, we obtain the following.

**Theorem 1.5.** *Let  $E$  be a vector bundle on  $Y$  of rank  $(d - 1)$  and degree  $e$ . If  $g$  is sufficiently large (depending on  $Y$  and  $E$ ), then for every line bundle  $L$  of degree  $d + g - e - 1$ , the Maroni*

locus  $M(E \otimes L) \subset H_{d,g}(Y)$  contains an irreducible component having the expected codimension  $h^1(\text{End } E)$ .

Theorem 1.5 is Corollary 3.11 in the main text.

Going further, it would be valuable to know whether all the components of  $M(E \otimes L)$  are of the expected dimension or, even better, if  $M(E \otimes L)$  is irreducible. The results of [6] imply irreducibility for  $Y = \mathbf{P}^1$  and some vector bundles  $E$ . But the question remains open in general.

The connection to cycles on  $H_{d,g}(Y)$  is through the fundamental class of the closure of  $M(E)$ . It would be interesting to know if this cycle has any distinguishing properties, such as rigidity or extremality, as is the case for the Maroni divisors for  $Y = \mathbf{P}^1$ , at least when  $d \leq 5$  [14].

We also draw the reader's attention to results, similar in spirit to Theorem 1.4, in work of Beauville, Narasimhan, and Ramanan [2]. The basic problem in this line of inquiry, first posed by Beauville, is to study not the pushforward of  $\mathcal{O}_X$  itself but the pushforwards of general line bundles on  $X$ .

The attempt at extending Theorem 1.2 to higher dimensional varieties  $Y$  presents interesting new challenges. We discuss them through some examples in § 4. As it stands, Question 1.1 and the analogue of Theorem 1.2 for higher dimensional varieties  $Y$  are false. We end the paper by posing slight modifications, for which we are still unable to find counterexamples.

**1.1. Strategy of proof.** The proof of Theorem 1.2 proceeds by degeneration. To help the reader, we first outline our approach to a weaker version of Theorem 1.2. In the weaker version, we consider not the vector bundle  $E$  itself, but its projectivization  $\mathbf{P}E$ , which we call the *Tschirnhausen scroll*. Recall that a branched cover with Gorenstein fibers  $\phi : X \rightarrow Y$  with Tschirnhausen bundle  $E$  factors through a *relative canonical embedding*  $\iota : X \hookrightarrow \mathbf{P}E$  (see [4]).

**Theorem 1.6.** *Let  $E$  be any vector bundle on a smooth projective curve  $Y$ . Then the scroll  $\mathbf{P}E$  is the Tschirnhausen scroll of a finite cover  $\phi : X \rightarrow Y$  with  $X$  smooth.*

Note that this already gives an affirmative answer to Question 1.1. The following steps outline a proof of Theorem 1.6 which parallels the proof of the stronger Theorem 1.2. We omit the details, since they are subsumed by the results in the sequel.

- (1) First consider the case

$$E = L_1 \oplus \cdots \oplus L_{d-1},$$

where the  $L_i$  are line bundles on  $Y$  whose degrees satisfy

$$\deg L_i \ll \deg L_{i+1}.$$

For such  $E$ , construct a nodal cover  $\phi : X \rightarrow Y$  such that  $E_\phi = E$ . For example, we may take  $X$  to be a nodal union of  $d$  copies of  $Y$ , each mapping isomorphically to  $Y$  under  $\phi$ , where the  $i$ th copy meets the  $(i+1)$ th copy along nodes lying in the linear series  $|L_i|$ .

- (2) The next step is to smooth out the nodes of  $X \subset \mathbf{P}E$ . As it stands, the normal bundle  $N_{X/\mathbf{P}E}$  is quite negative; in fact simple examples show that the nodes of  $X$  are usually impossible to smooth out. Fixing this negativity is the most crucial step.
- (3) Attach several rational normal curves to  $X$  as follows. Given a general point  $y \in Y$ , the  $d$  points  $\phi^{-1}(y) \subset \mathbf{P}E_y \simeq \mathbf{P}^{d-2}$  are in general linear position, and therefore they lie on many smooth rational normal curves  $R_y \subset \mathbf{P}E_y$ . Choose a large subset  $S \subset Y$ , and attach general rational normal curves  $R_y$  for each  $y \in S$  to  $X$ , obtaining a new nodal curve  $\tilde{X} \subset \mathbf{P}E$ .

- (4) Next, see that the new normal bundle  $N_{\tilde{X}/\mathbf{P}E}$  is sufficiently positive. In particular,  $\tilde{X}$  is the flat limit of a family of smooth, relatively-canonically embedded curves  $X_t \subset \mathbf{P}E$ . Furthermore, the higher cohomology of  $N_{X_t/\mathbf{P}E}$  vanishes.
- (5) Deduce the result for arbitrary bundles  $E$  as follows.
  - (a) See that every vector bundle  $E$  degenerates isotrivially to a direct sum of line bundles  $L_i$  of the form needed above.
  - (b) Consider the map  $\phi \rightarrow \mathbf{P}E_\phi$  be the map from the moduli stack of branched covers of  $Y$  to the moduli stack of projective bundles on  $Y$ . The positivity of  $N_{X_t/\mathbf{P}E}$  implies that this map is smooth at  $X_t \rightarrow Y$ .
  - (c) Using the openness of smooth maps and the previous two steps, conclude that every projective bundle arises as a Tschirnhausen scroll for a smooth cover.

To have control over the vector bundle  $E$  itself, and not just its projectivization, we consider the *canonical affine embedding* of  $X$  in the total space of  $E$ . However, to attach rational normal curves, it is essential to have a projective space bundle. Therefore, we consider the projective closure  $P := \mathbf{P}(E^\vee \oplus \mathcal{O}_Y)$  of the total space of  $E$ . Let  $H := \mathbf{P}(E^\vee \oplus \mathcal{O}_Y) \setminus E$  be the divisor of hyperplanes at infinity. The proof of Theorem 1.2 involves carrying out the steps outlined above for the embedding  $X \rightarrow P$  relative to the divisor  $H$ .

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**1.3. Conventions.** All schemes are finite type over an algebraically closed field  $k$  of characteristic 0 (or of characteristic larger than the degree  $d$  of the covers we consider). The projectivization  $\mathbf{P}V$  of a vector bundle  $V$  refers to the space of 1-dimensional quotients of  $V$ . We identify vector bundles with their sheaves of sections. An injection of vector bundles is understood as an injection of the corresponding locally free sheaves.

## 2. VECTOR BUNDLES, THEIR INFLATIONS, AND DEGENERATIONS

Let  $E$  be a vector bundle on  $Y$ . A *degree  $n$  inflation* of  $E$  is a vector bundle  $\tilde{E}$  along with an injective map  $E \rightarrow \tilde{E}$  whose cokernel is finite of length  $n$ . If  $E \rightarrow \tilde{E}$  is an inflation, then the dual bundle  $\tilde{E}^\vee$  is a sub-bundle of  $E^\vee$  and the quotient is finite of length  $n$ . Thus, a degree  $n$  inflation of  $E$  is equivalent to a sub-bundle of  $E^\vee$  of co-length  $n$ , which in turn is equivalent to a quotient of  $E^\vee$  of length  $n$ . Therefore, we can identify the moduli space of length  $n$  inflations of  $E$  with the quot scheme  $\text{Quot}(E^\vee, n)$ . It is easy to see that  $\text{Quot}(E^\vee, n)$  is smooth and connected, in particular irreducible. Therefore, it makes sense to talk about a general length  $n$  inflation of  $E$ .

**Proposition 2.1.** *Let  $E$  be a vector bundle on  $Y$ . For a sufficiently large  $n$ , a general length  $n$  inflation  $E \rightarrow \tilde{E}$  satisfies  $H^1(\tilde{E}) = 0$ .*

*Proof.* By Serre vanishing, we have  $H^1(E \otimes L) = 0$  for a very ample line bundle  $L$  on  $Y$ . Set  $N = \text{rk } E \cdot \deg L$ . Let  $n \geq N$  and let  $E \otimes L \rightarrow \tilde{E}$  be a general length  $(N - n)$  inflation. Then  $E \rightarrow \tilde{E}$  is a length  $n$  inflation. We have an exact sequence

$$(2.1) \quad 0 \rightarrow E \otimes L \rightarrow \tilde{E} \rightarrow Q \rightarrow 0.$$

Since  $H^1(E \otimes L) = 0$ , the long exact sequence on cohomology associated to (2.1) implies that  $H^1(\tilde{E}) = 0$ . Since the space of length  $n$  inflations of  $E$  is irreducible, we conclude that  $H^1(\tilde{E}) = 0$  for a general length  $n$  inflation  $E \rightarrow \tilde{E}$  for any  $n \geq N$ .  $\square$

*Remark 2.2.* Suppose  $E \rightarrow \tilde{E}$  is an inflation and  $H^1(E) = 0$ . Then the long exact sequence associated to

$$0 \rightarrow E \rightarrow \tilde{E} \rightarrow Q \rightarrow 0$$

shows that we also have  $H^1(\tilde{E}) = 0$ .

*Remark 2.3.* Consider a degree 1 inflation  $E \rightarrow \tilde{E}$ . Suppose the cokernel is supported at a point  $p$ . The dual map  $\tilde{E}^\vee \rightarrow E^\vee$  drops rank by 1 at  $p$ , and therefore the image of  $\tilde{E}^\vee|_p$  in  $E^\vee|_p$  is a hyperplane. Conversely, a degree 1 inflation of  $E$  is specified by a point  $p$  and a hyperplane of  $E^\vee|_p$ . In this case,  $E$  and  $\tilde{E}$  are often said to be related by an elementary transformation.

*Remark 2.4.* The context in which we use inflations is the following. Let  $P$  be a smooth variety. Let  $R, S \subset P$  be curves that intersect at a point  $p$  so that their union  $Z$  has a simple node at  $p$ . Then we have the exact sequence

$$0 \rightarrow N_{R/P} \rightarrow N_{Z/P}|_R \rightarrow k_p \rightarrow 0.$$

That is, the bundle  $N_{Z/P}|_R$  is a degree 1 inflation of  $N_{R/P}$ . The hyperplane of  $N_{R/P}^\vee|_p$  that specifies this inflation is the kernel of the map

$$N_{R/P}^\vee|_p \rightarrow k$$

defined as the composite

$$N_{R/P}^\vee|_p \xrightarrow{d} \Omega_P|_p \rightarrow k,$$

where the last map is the contraction with a non-zero vector in  $T_p S$ . In particular, if the point  $p \in R$  and the image of  $T_p S$  in  $N_{R/P}|_p$  are both general, then  $N_{Z/P}|_R$  is a general degree 1 inflation of  $N_{R/P}$ .

We say that a bundle  $E$  *isotrivially degenerates* to a bundle  $E_0$  if there exists a pointed smooth (not necessarily projective) curve  $(\Delta, 0)$  and a  $\Delta$ -flat bundle  $\mathcal{E}$  on  $Y \times \Delta$  such that  $\mathcal{E}_{Y \times \{0\}} \cong E_0$  and  $\mathcal{E}_{Y \times \{t\}} \cong E$  for every  $t \in \Delta \setminus \{0\}$ .

**Proposition 2.5.** *Let  $E$  a vector bundle on  $Y$ , and  $N$  a non-negative integer. Then  $E$  isotrivially degenerates to a vector bundle  $E_0$  of the form*

$$E_0 = L_1 \oplus \cdots \oplus L_r,$$

where the  $L_i$  are line bundles and  $\deg L_i + N \leq \deg L_{i+1}$ .

For the proof of Proposition 2.5, we need a lemma.

**Lemma 2.6.** *There exists a filtration*

$$E = F_0 \supset F_1 \supset \cdots \supset F_{r-1} \supset F_r = 0,$$

satisfying the following properties.

- (1) For every  $i \in \{0, \dots, r-1\}$ , the subquotient  $F_i/F_{i+1}$  is a line bundle.
- (2) Set  $L_i = F_i/F_{i+1}$  for  $i \in \{1, \dots, r-1\}$  and  $L_r = F_0/F_1$ . For every  $i \in \{1, \dots, r-1\}$ , we have

$$\deg L_i + N \leq \deg L_{i+1}.$$

*Proof.* The statement is vacuous for  $r = 0$  and 1. So assume  $r \geq 2$ . Note that if  $F_\bullet$  is a filtration of  $E$  satisfying the two conditions, and if  $L$  is invertible, then  $F_\bullet \otimes L$  is such a filtration of  $E \otimes L$ . Therefore, by twisting by a line bundle of large degree if necessary, we may assume that  $\deg E \geq 0$ .

Let us construct the filtration from right to left. Let  $L_{r-1} \subset E$  be a line bundle with  $\deg L_{r-1} \leq -N$  and with a locally free quotient. Set  $F_{r-1} = L_{r-1}$ . Next, let  $L_{r-2} \subset E/F_{r-1}$  be a line bundle with  $\deg L_{r-2} \leq \deg L_{r-1} - N$  and with a locally free quotient. Let  $F_{r-2} \subset E$  be the preimage of  $L_{r-2}$ . Continue in this way. More precisely, suppose that we have constructed

$$F_j \supset F_{j+1} \supset \cdots \supset F_{r-1} \supset F_r = 0$$

such that  $L_i = F_i/F_{i+1}$  satisfy

$$\deg L_i \leq \deg L_{i+1} - N,$$

and suppose  $j \geq 2$ . Then let  $L_{j-1} \subset E/F_j$  be a line bundle with  $\deg L_{j-1} \leq \deg L_j - N$  with a locally free quotient. Let  $F_{j-1} \subset E$  be the preimage of  $L_{j-1}$ . Finally, set  $F_0 = E$ .

Condition (1) is true by design. Condition (2) is true by design for  $i \in \{1, \dots, r-2\}$ . For  $i = r-1$ , note that  $\deg L_{r-1} \leq -N$  by construction. On the other hand, we must have  $\deg L_r \geq 0$ . Indeed, we have  $\deg E \geq 0$  but every sub-quotient of  $F_\bullet$  except  $F_0/F_1$  has negative degree. Therefore, condition (2) holds for  $i = r-1$  as well.  $\square$

*Proof of Proposition 2.5.* Let  $F_\bullet$  be a filtration of  $E$  satisfying the conclusions of Lemma 2.6. It is standard that a coherent sheaf degenerates isotrivially to the associated graded sheaf of its filtration. The construction goes as follows. Consider the  $\mathcal{O}_Y[t]$ -module

$$\bigoplus_{n \in \mathbb{Z}} t^{-n} F_n,$$

where  $F_n = 0$  for  $n > r$  and  $F_n = E$  for  $n < 0$ . The corresponding sheaf  $\mathcal{E}$  on  $Y \times \mathbb{A}^1$  is coherent,  $k[t]$ -flat, satisfies  $\mathcal{E}_{Y \times \{t\}} \cong E$  for  $t \neq 0$ , and  $\mathcal{E}_{Y \times \{0\}} \cong L_1 \oplus \cdots \oplus L_r$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

**3.1. The split case.** As a first step, we treat the case of a suitable direct sum of line bundles and allow the source curve to be singular.

**Proposition 3.1.** *Let  $E = L_1 \oplus \cdots \oplus L_r$ , where the  $L_i$  are line bundles on  $Y$  with  $\deg L_1 \geq 2g_Y - 1$  and  $\deg L_{i+1} \geq \deg L_i + (2g_Y - 1)$  for  $i \in \{1, \dots, r-1\}$ . There exists a nodal curve  $X$  and a finite flat map  $\phi : X \rightarrow Y$  of degree  $d = r+1$  such that*

- (1) *we have  $E_\phi \cong E$ , and*
- (2) *the normalization  $X^\vee$  is isomorphic to  $d$  disjoint copies of  $Y$ .*

The proof is inductive, based on the following “pinching” construction. Let  $\psi : Z \rightarrow Y$  be a finite cover of degree  $d-1$ . Let  $X$  be the reducible nodal curve  $Z \cup Y$ , where  $Z$  and  $Y$  are attached nodally at distinct points. We have a finite flat map  $\phi : X \rightarrow Y$  that restricts to  $\psi$  on  $Z$  and is identity on  $Y$ . Let  $D \subset Y$  be the preimage of the nodes.

**Lemma 3.2.** *In the setup above, we have an exact sequence*

$$0 \rightarrow E_\psi \rightarrow E_\phi \rightarrow \mathcal{O}_Y(D) \rightarrow 0.$$

*Proof.* The closed embedding  $Z \rightarrow X$  gives a surjection

$$\phi_* \mathcal{O}_X \rightarrow \psi_* \mathcal{O}_Z$$

whose kernel is easily seen to be  $\mathcal{O}_Y(-D)$ . Factoring out the  $\mathcal{O}_Y$  summand from both sides and taking duals yields the claimed exact sequence.  $\square$

*Proof of Proposition 3.1.* We use induction on  $r$ , starting with the base case  $r = 0$ , which is vacuous.

By the inductive hypothesis, we may assume that there exists a nodal curve  $Z$  and a finite cover  $\psi: Z \rightarrow Y$  of degree  $d - 1$  such that  $E_\psi \cong L_2 \oplus \cdots \oplus L_r$  and  $Z^\vee$  is a disjoint union of  $d - 1$  copies of  $Y$ . Let  $X = Z \cup Y \rightarrow Y$  be a cover of degree  $d$  obtained from  $Z \rightarrow Y$  by a pinching construction such that  $\mathcal{O}_Y(D) = L_1$ . Then  $X^\vee$  is a disjoint union of  $d$  copies of  $Y$ . By Lemma 3.2, we get an exact sequence

$$(3.1) \quad 0 \rightarrow L_2 \oplus \cdots \oplus L_r \rightarrow E_\phi \rightarrow L_1 \rightarrow 0.$$

But we have  $\text{Ext}^1(L_1, L_i) = H^1(L_i \otimes L_1^\vee) = 0$  since  $\deg(L_i \otimes L_1^\vee) \geq 2g_Y - 1$ . Therefore, the sequence (3.1) is split, and we get  $E_\phi = L_1 \oplus \cdots \oplus L_r$ . The induction step is then complete.  $\square$

**3.2. Smoothing out.** We now come to the key step of the proof. This step allows us to pass from singular covers to smooth covers and from particular vector bundles to their deformations.

Let  $X$  be a nodal curve,  $\phi: X \rightarrow Y$  a finite flat morphism of degree  $d$ , and  $E$  the associated Tschirnhausen bundle. The inclusion  $E^\vee \rightarrow \phi_* \mathcal{O}_X$  induces a surjection  $\text{Sym}^* E^* \rightarrow \phi_* \mathcal{O}_X$ . Taking the relative spectrum gives an embedding of  $X$  in the total space of the vector bundle associated to  $E$  (which we also denote by  $E$ ). We call  $X \subset E$  the *canonical affine embedding*.

**Proposition 3.3 (Key).** *There exists a line bundle  $L$  on  $Y$ , a smooth curve  $X'$ , and a finite morphism  $\phi': X' \rightarrow Y$  such that the following hold.*

- (1) *The Tschirnhausen bundle of  $\phi'$  is  $E \otimes L$ .*
- (2) *We have  $H^1(X', N_{X'/E'}) = 0$ , where  $X' \subset E'$  is the canonical affine embedding.*

*Furthermore, there exists an  $n$  (depending on  $X$ ), such that the above holds for any  $L$  of degree at least  $n$ .*

The crucial idea in the proof of Proposition 3.3 is the following construction. Let  $S \subset Y$  be a finite set such that  $X \rightarrow Y$  is étale over all points of  $S$ . Consider the compactification  $P = \mathbf{P}(E^\vee \oplus \mathcal{O}_Y)$  of  $E$ . Let  $H \cong \mathbf{P}E^\vee \subset P$  be the family of hyperplanes at infinity, where the embedding  $H \subset P$  is defined by the projection  $E^\vee \oplus \mathcal{O}_Y \rightarrow E^\vee$ . The complement of  $H \subset P$  is the vector bundle  $E$ . For  $y \in S$ , the set  $X_y \subset P_y \cong \mathbf{P}^{d-1}$  consists of  $d$  points in general position. Therefore, there exists a smooth rational normal curve  $R_y$  in  $P_y$  containing  $X_y$ . Let  $\tilde{P} \rightarrow P$  be the blow up at  $\bigsqcup_{y \in S} H_y$ . Denote by the same symbol  $R_y$  the proper transform of  $R_y$  in  $\tilde{P}$ , and by  $\tilde{H}$  the proper transform of  $H$  in  $\tilde{P}$ .

The fiber of  $\tilde{P} \rightarrow Y$  over  $y \in S$  consists of two components. One is the exceptional divisor  $E_y$  of the blow-up. The second is the proper transform  $Q_y$  of  $P_y$ , which is a copy of  $P_y$ . The two components intersect transversally along a  $\mathbf{P}^{d-2}$ . See Figure 1 for a picture of this construction.

Set  $Z = X \cup_{y \in S} R_y$ . The proof of Proposition 3.3 proceeds through the following result.

**Proposition 3.4.** *For large enough  $n$ , a general choice of  $S \subset Y$  of size  $n$ , and a general choice of rational normal curves  $R_y$  for  $y \in S$ , the curve  $Z$  is unobstructed in  $\tilde{P}$  and is a flat limit of smooth curves in  $\tilde{P}$ .*

*Furthermore, if  $n$  is large enough, the set  $S$  can be chosen so that  $\mathcal{O}_Y(S)$  is isomorphic to any prescribed line bundle of degree  $n$ .*

We need several preparatory lemmas for the proof of Proposition 3.4. First, we set some notation. Denote the normal bundle  $N_{Z/\tilde{P}}$  by  $N$  for brevity. Let  $\nu: Z^\vee \rightarrow Z$  be the normalization. Note that  $Z^\vee$  is the disjoint union of the components  $X_1, \dots, X_l$  of the normalization  $X^\vee$  of  $X$

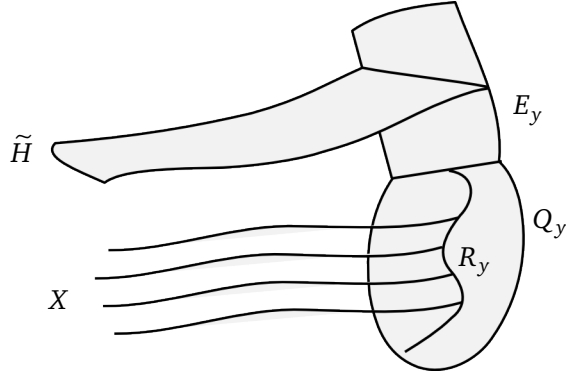


FIGURE 1. Attaching rational normal curves to  $X$  to make the normal bundle positive

and the rational curves  $R_y$  for  $y \in S$ . Denote by  $v_X: X^\vee \rightarrow Z$ ,  $v_i: X_i \rightarrow Z$ , and  $v_y: R_y \rightarrow Z$  the natural maps. Let  $\gamma \subset Z$  be the set of nodes of  $X$  and  $\Gamma \subset X^\vee$  its preimage. Note that every point of  $\gamma$  has two preimages in  $\Gamma$ . Set  $\delta_y = R_y \cap X$  and  $\delta = \bigcup_{y \in S} \delta_y$ . Then the singular set of  $Z$  is  $\gamma \cup \delta$ .

Let  $y$  be a point in  $S$ .

**Lemma 3.5.** *The restriction of  $N$  to  $R_y$  is isomorphic to  $\mathcal{O}(d+1)^{d-2} \oplus \mathcal{O}(1)$ , and the  $\mathcal{O}(d+1)^{d-2}$  summand is the image of the natural map*

$$N_{R_y/Q_y} \rightarrow N|_{R_y}.$$

*Proof.* In the proof, we drop the subscript  $y$  from  $R_y$  and  $Q_y$ . First, note that  $N|_R$  is a vector bundle of rank  $(d-1)$  and degree  $(d-2)(d+1)+1$ . The map  $N_{R/Q} \rightarrow N|_R$  is the composite

$$N_{R/Q} \rightarrow N_{R/\tilde{P}} \rightarrow N_{Z/\tilde{P}}|_R = N|_R.$$

Using that  $X$  is transverse to  $Q$ , a simple local computation shows that the injection  $N_{R/Q} \rightarrow N|_R$  remains an injection when restricted to any point of  $R$  (see Remark 2.4). Since  $R \subset Q \cong \mathbf{P}^{d-1}$  is a rational normal curve, we know that  $N_{R/Q} \cong \mathcal{O}(d+1)^{d-2}$ . We thus get an exact sequence

$$0 \rightarrow \mathcal{O}(d+1)^{d-2} \rightarrow N|_R \rightarrow \mathcal{O}(1) \rightarrow 0.$$

Since  $\text{Ext}^1(\mathcal{O}(1), \mathcal{O}(d+1)) = 0$ , this sequence splits.  $\square$

We call the  $\mathcal{O}(d+1)^{d-2}$  the *vertical* summand of  $N|_{R_y}$ . Let  $V \subset N|_\delta$  be the image of all the vertical summands of  $R_y$  for  $y \in S$ . Set  $F = N|_\delta/V$ . Denote by  $V_y$  and  $F_y$  the restrictions of  $V$  and  $F$  to  $\delta_y$ , respectively.

**Lemma 3.6.** *We have an exact sequence*

$$0 \rightarrow N_{X/\tilde{P}} \rightarrow N|_X \rightarrow F \rightarrow 0.$$

*Proof.* We have an injective map of vector bundles  $N_{X/\tilde{P}} \rightarrow N|_X$  that drops rank by 1 at every point of  $\delta$  (see Remark 2.4). Let  $p \in \delta$  lie over  $y \in Y$ . To show that the cokernel of  $N_{X/\tilde{P}} \rightarrow N|_X$  is  $F$ , we must show that the image of  $N_{X/\tilde{P}}|_p \rightarrow N|_p$  is  $V|_p$ . But this follows from the following



commutative diagram

$$\begin{array}{ccccc} T_{Q_y}|_p & \longrightarrow & N_{R_y/Q_y}|_p & \longrightarrow & 0 \\ \parallel & & \downarrow & & \\ N_{X/\tilde{p}}|_p & \longrightarrow & N|_p & & \end{array}$$

□

Let  $X_1, \dots, X_\ell$  be the components of the normalization  $X^\vee$  of  $X$ , and let  $\nu_i: X_i \rightarrow X$  be the induced maps. Let  $p \in \delta$  be a point lying on  $X_i$ . Let  $N_i$  be the kernel of the map  $N|_X \rightarrow F_{\delta \setminus \{p\}}$ . Then we have an exact sequence

$$0 \rightarrow \nu_i^* N_{X/\tilde{p}} \rightarrow N_i \rightarrow F_p \rightarrow 0.$$

In other words,  $N_i$  is an inflation of  $\nu_i^* N_{X/\tilde{p}}$  at  $p$ . Furthermore, if the tangent line  $T_p R_y \subset T_p Q_y$  is general, then  $N_i$  is a general inflation of  $\nu_i^* N_{X/\tilde{p}}$  at  $p$ . Note that we have an injection  $\nu_i^* N_i \rightarrow \nu_i^* N$ , which is an isomorphism except at the points of  $\delta \setminus \{p\}$ .

**Lemma 3.7.** *If the size of  $S$  is large, its points are general, and the rational normal curves  $R_y$  are general, then we have  $H^1(X_i, \nu_i^* N) = 0$ .*

*Proof.* The preceding analysis show that  $\nu_i^* N$  contains a general degree  $|S|$  inflation of  $\nu_i^* N_{X/\tilde{p}}$ . The lemma follows from Proposition 2.1. □

Thanks to Lemma 3.5 and Lemma 3.7, the restriction of  $N$  to the components of the normalization of  $Z$  has no higher cohomology. This is necessary, but not sufficient to conclude that  $H^1(N) = 0$ . To get the latter, we need a more careful analysis.

Recall that  $\Gamma \subset X^\vee$  is the preimage of the singular set  $\gamma$  of  $X$ . Set  $M = \nu_X^* N_{X/\tilde{p}}(-\Gamma)$ . Note that the natural map  $M \rightarrow \nu_X^* N_{X/\tilde{p}}$  is an isomorphism at the points of  $\delta$ . From now on, we identify  $M|_\delta$  and  $N|_\delta$ .

The surjection  $M \rightarrow M|_\delta$  gives a surjection

$$\phi_* M \rightarrow \phi_* (M|_\delta).$$

Set  $R = \cup_{y \in S} R_y$ . The surjection  $N|_R \rightarrow M|_\delta$  gives a map

$$\phi_* (N|_R) \rightarrow \phi_* (M|_\delta).$$

Let  $W$  be the cokernel; it is supported on  $S$ . For  $y \in S$ , we have the diagram with exact rows

$$(3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \phi_* \left( \mathcal{O}_{R_y} (d+1)^{d-2} \right) & \longrightarrow & \phi_* \left( N|_{R_y} \right) & \longrightarrow & \phi_* \left( \mathcal{O}_{R_y} (1) \right) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \phi_* \left( V|_{\delta_y} \right) & \longrightarrow & \phi_* \left( M|_{\delta_y} \right) & \longrightarrow & \phi_* \left( F_y \right) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & W_y & \xlongequal{\quad} & W_y. \end{array}$$

In particular, we get  $W_y \cong k^{d-2}$ . Let  $K$  be the kernel of the surjection  $\phi_* M \rightarrow W$ . Then  $K$  is a vector bundle of rank  $d(d-1)$  on  $Y$ .

**Proposition 3.8.** *If the size of  $S$  is large, its points are general, and the rational normal curves  $R_y$  are general, then we have  $H^1(Y, K) = 0$ .*

*Proof.* The proof is by degeneration. For simplicity, assume first that  $S$  consists of a single point  $y$ . Take a point  $p \in \delta_y$ . Consider a one parameter degeneration of  $R_y$  to a reducible nodal curve  $R' = R'_1 \cup R'_2$  contained in  $Q_y$  and containing  $\delta_y$  of the following form:  $R'_1$  is a line containing  $p$ , and  $R'_2$  is a smooth rational curve of degree  $(d-2)$  containing  $\delta_y \setminus \{p\}$ , with  $R'_1$  and  $R'_2$  meeting nodally at one point away from  $\delta_y$ . More formally, let  $(T, 0)$  be a smooth pointed curve, and  $S \subset Q_y \times T$  a smooth surface containing  $\delta_y \times T$  such that  $S_0 = R'$  as described above and  $S_t \subset Q_y$  is a smooth rational normal curve for  $t \neq 0$ . We may assume that  $T_p R' = T_p R'_1$  is a general line in  $T_p Q_y$ . Set  $Z = (X \times T) \cup S \subset \tilde{P} \times T$ . Let  $\mathcal{N} = N_{Z/(\tilde{P} \times T)}$  and  $\mathcal{M} = \mathcal{N} \otimes \mathcal{O}_S(-\Gamma \times T)$ .

Define  $\mathcal{L}$  by the exact sequence

$$0 \rightarrow N_{S/(Q_y \times T)} \rightarrow N_{Z/(\tilde{P} \times T)}|_S \rightarrow \mathcal{L} \rightarrow 0.$$

For a general  $t \in T$ , we have  $\mathcal{L}_t \cong \mathcal{O}(1)$ . For  $t = 0$ , we have  $\mathcal{L}_0|_{R'_1} \cong \mathcal{O}$  and  $\mathcal{L}_0|_{R'_2} \cong \mathcal{O}(1)$ . Set  $\mathcal{F} = \mathcal{L}|_{\delta \times T}$ . We have a surjection  $\mathcal{M} \rightarrow \mathcal{F}$ .

Consider  $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_S(R'_2)$ . Then  $\mathcal{L}'$  is isomorphic to  $\mathcal{L}$  away from  $t = 0$ . On  $t = 0$ , we have  $\mathcal{L}'|_{R'_1} \cong \mathcal{O}(1)$  and  $\mathcal{L}'|_{R'_2} \cong \mathcal{O}$ . Clearly,  $\mathcal{L}'|_{\delta \times T}$  is isomorphic to  $\mathcal{L}|_{\delta \times T}$ . Identify them.

Define  $\mathcal{W}$  by the sequence

$$\phi_* \mathcal{L}' \rightarrow \phi_* \mathcal{F} \rightarrow \mathcal{W} \rightarrow 0.$$

Then  $\mathcal{W}$  is a  $T$ -flat  $Y \times T$  module supported on  $S \times T$ . Let  $\mathcal{K}$  be the kernel of the surjection  $\phi_* \mathcal{M} \rightarrow \mathcal{W}$ . Over a generic  $t \in T$ , the bundle  $\mathcal{K}|_t$  on  $Y$  is simply the bundle  $K$ . Let us analyze  $\mathcal{K}|_0$ . Observe that the image of  $\phi_* \mathcal{L}'|_0 \rightarrow \phi_* \mathcal{F}|_0$  contains the global section of  $\mathcal{F}|_0$  that is non-zero at  $p$  and zero at the other points of  $\delta$ . Said differently,  $\mathcal{W}_0$  is a further quotient of  $\phi_* (\mathcal{F}|_{(\delta \setminus \{p\}) \times \{0\}})$ .

Define  $M^+$  on  $X^\vee$  by the sequence

$$0 \rightarrow M^+ \rightarrow \nu_X^* M \rightarrow \mathcal{F}|_{(\delta \setminus \{p\}) \times \{0\}} \rightarrow 0.$$

Then  $M^+$  is isomorphic to  $\nu_X^* N_{X/\tilde{P}}(-\Gamma)$  on all components of  $X^\vee$  except the one containing  $p$ . On the component containing  $p$ , it is a degree 1 inflation (see the discussion after Lemma 3.6) of  $\nu_i^* N_{X/\tilde{P}}(-\Gamma)$  at  $p$ . Furthermore, since  $T_p R'$  is a general line in  $T_p Q_y$ , it is a general such inflation. Since  $\mathcal{W}_0$  is a quotient of  $\phi_* (\mathcal{F}|_{(\delta \setminus \{p\}) \times \{0\}})$ , we get an injection  $\phi_* M^+ \rightarrow \mathcal{K}|_0$ .

Now consider  $S \subset Y$  of size  $n$ , say  $S = \{y_1, \dots, y_n\}$ . Pick  $p_{ij} \in X$  over  $y_i$  such that a general inflation of  $\nu_X^* N_{X/\tilde{P}}(-\Gamma)$  at the points  $p_{ij}$  has vanishing  $H^1$ . By considering an  $n$ -parameter degeneration of the  $n$  rational normal curves  $R_y$  for  $y \in S$ , we obtain a family of bundles  $\mathcal{K}$  whose generic fiber  $\mathcal{K}|_t$  agrees with  $K$  and whose special fiber  $\mathcal{K}|_0$  contains  $\phi_* M^+$ , where  $M^+$  is a general inflation of  $\nu_X^* N_{X/\tilde{P}}(-\Gamma)$  at the points  $p_{ij}$ . It follows that  $H^1(Y, \mathcal{K}|_0) = 0$ , and hence  $H^1(Y, K) = 0$  by semicontinuity.  $\square$

We now have all the tools to prove Proposition 3.4.

*Proof of Proposition 3.4.* Retain the notation introduced so far in § 3.2.

We have the exact sequence

$$0 \rightarrow N \rightarrow \nu_* \nu^* N \rightarrow N|_{\gamma \cup \delta} \rightarrow 0.$$

The long exact sequence in cohomology gives

$$H^0(\nu^* N) \rightarrow H^0(N|_{\gamma \cup \delta}) \rightarrow H^1(N) \rightarrow H^1(\nu^* N) \rightarrow 0.$$

By Lemma 3.5, we know that  $H^1(\nu_y^* N) = 0$ . By Lemma 3.6, we know that  $H^1(\nu_X^* N) = 0$ . Therefore, we get  $H^1(\nu^* N) = 0$ .

We now show that  $H^0(v^*N) \rightarrow H^0(N|_{\gamma \cup \delta})$  is surjective. First, by the definition of  $W$ , we have the exact sequence

$$(3.3) \quad \bigoplus_{y \in S} H^0(v_y^*N) \rightarrow H^0(N|_{\delta}) \rightarrow H^0(W) \rightarrow 0.$$

Second, the sequence

$$0 \rightarrow K \rightarrow \phi_* v_X^* N_{Z/\tilde{P}} \rightarrow \phi_* (v_X^* N_{Z/\tilde{P}}|_{\Gamma}) \oplus W \rightarrow 0,$$

along with Proposition 3.8, gives a surjection

$$(3.4) \quad H^0(v_X^* N) \rightarrow H^0(v^* N|_{\Gamma}) \oplus H^0(W).$$

By combining (3.3) and (3.4), we get a surjection

$$(3.5) \quad H^0(v^* N) \rightarrow H^0(v^* N|_{\Gamma}) \oplus H^0(N|_{\delta}).$$

Since  $H^0(v^* N|_{\Gamma}) \rightarrow H^0(N|_{\gamma})$  is surjective, we get that  $H^0(v^* N) \rightarrow H^0(N|_{\gamma \cup \delta})$  is surjective. We conclude that  $H^1(N) = 0$ .

Consider the effect of enlarging  $S$  to  $S^+ = S \cup \{y\}$  where  $y \in Y \setminus S$  is a point over which  $X \rightarrow Y$  is étale. Denote by  $Z^+$ ,  $N^+$ ,  $F^+$ , and  $V^+$  the analogues of  $Z$ ,  $N$ ,  $F$ , and  $V$  for  $S^+$ . Note that  $Z^+ = Z \cup R_y$ . Recall that  $N^+|_{\delta_y} = V_y^+ \oplus F_y^+$ . The exact sequence

$$0 \rightarrow N \rightarrow N^+|_Z \rightarrow F_y^+ \rightarrow 0$$

shows that  $H^1(N^+|_Z) = 0$  and  $H^0(N^+|_Z) \rightarrow H^0(F_y^+)$  is surjective. Since  $H^1(N^+|_{R_y}) = 0$  and  $H^0(N^+|_{R_y}) \rightarrow H^0(V_y^+)$  is surjective, we deduce that  $H^1(N^+) = 0$ . Therefore, enlarging  $S$  retains the vanishing  $H^1(N) = 0$ .

Since  $H^1(N) = 0$ , the Hilbert scheme of  $\tilde{P}$  is smooth at  $[Z]$ . In particular, every first order deformation of  $Z \subset \tilde{P}$  extends. To show that  $Z$  is the limit of smooth curves, it suffices to show that for every node  $p \in Z$ , the natural map  $N \rightarrow k_p$  is surjective on global sections, where  $k_p$  is a skyscraper sheaf at  $p$ . The surjection  $N|_p \rightarrow k_p$  is a part of the exact sequence

$$0 \rightarrow T_p \tilde{P} / T_p Z \rightarrow N|_p \rightarrow k_p \rightarrow 0.$$

In particular, for  $p \in \delta$  the map  $N \rightarrow k_p$  is the same as the map  $N \rightarrow F_p$ .

Consider a node  $p \in \gamma$ . The exact sequence

$$0 \rightarrow N \otimes I_{\Gamma} \rightarrow v_* v^* N \rightarrow v_* (v^* N|_{\Gamma}) \oplus N|_{\delta} \rightarrow 0$$

and the surjection (3.5) implies  $H^1(N \otimes I_{\Gamma}) = 0$ . Therefore,  $H^0(N) \rightarrow H^0(N|_p)$  and hence  $H^0(N) \rightarrow H^0(k_p)$  are surjective for all  $p \in \gamma$ .

Next, consider a node  $p \in \delta_y$ . Let  $S^- = S \setminus \{y\}$ . Denote by  $Z^-$ ,  $N^-$ ,  $F^-$ , and  $V^-$  the analogues of  $Z$ ,  $N$ ,  $F$ , and  $V$  for  $S^-$ . Let  $\mu: Z^- \sqcup R_y \rightarrow Z$  be the natural map, which is the normalization of the nodes  $\delta_y$  of  $Z$ . We have a surjection  $H^0(R_y, \mu^* N|_{R_y}) \rightarrow H^0(F_p)$ . If  $S$  is large enough, we may assume that we already have  $H^1(N^-) = 0$ . Then we have a surjection  $H^0(Z^-, \mu^* N|_{Z^-}) \rightarrow H^0(F_y)$ . Combining the two, we see that we have a surjection  $H^0(N) \rightarrow H^0(F_p)$ .

Finally, assume that  $n$  is large enough so that the conclusions above hold for an  $S$  of size  $n - 2g_Y$ . Then we may enlarge  $S$  to a set  $S^+$  by adding an appropriate set of  $2g_Y$  points so that the same conclusions hold and  $\mathcal{O}_Y(S^+)$  is isomorphic to a given line bundle of degree  $n$ .  $\square$

We finally prove the key proposition.

*Proof of Proposition 3.3.* By Proposition 3.4, there exists a family of smooth curve in  $\tilde{P}$  whose flat limit is  $Z$ . Let  $X'$  be a general member of such a family. This curve satisfies the following conditions

- (1)  $\deg(X' \cdot E_y) = d - 1$  for all  $y \in S$ ,
- (2)  $\deg(X' \cdot Q_y) = 1$  for all  $y \in S$ ,
- (3)  $X' \cap \tilde{H} = \emptyset$ ,
- (4)  $g(X') = g(X) + n(d - 1)$ ,
- (5)  $H^1(N_{X'/\tilde{P}}) = 0$ .

Let  $\tilde{P} \rightarrow P'$  be the blowing down of all the  $Q_y$  for  $y \in S$ . Then  $P' \rightarrow Y$  is a  $\mathbf{P}^{d-1}$  bundle and the map  $X' \rightarrow P'$  is an embedding. Similarly,  $\tilde{H} \rightarrow P'$  is also an embedding and its complement is the total space of the vector bundle  $E' = E \otimes \mathcal{O}_Y(S)$ . Note that  $X'$  and  $\tilde{H}$  remain disjoint in  $P'$ , and hence we get an embedding  $X' \subset E'$ .

We claim that  $X' \subset E'$  is the canonical affine embedding. Consider the natural map

$$\phi_* \mathcal{O}_{P'}(\tilde{H}) = \mathcal{O}_Y \oplus E'^\vee \rightarrow \phi_* \mathcal{O}_{X'}.$$

The vector bundles  $\mathcal{O}_Y \oplus E'^\vee$  and  $\phi_* \mathcal{O}_{X'}$  have the same degree and rank, and the above map between them is an isomorphism at a generic point of  $Y$ . Therefore, it is an isomorphism. As a result, we get that  $E'$  is the Tschirnhausen bundle of  $X'$ , and the embedding  $X' \rightarrow E'$  is the canonical affine embedding.

Next, note that we have an injection of vector bundles

$$N_{X'/\tilde{P}} \rightarrow N_{X'/E'}$$

with finite quotient (supported on  $\bigcup_{y \in S} X' \cap E_y$ ). Since  $H^1(N_{X'/\tilde{P}}) = 0$ , we deduce that  $H^1(N_{X'/E'}) = 0$ .

Finally, by the last assertion of Proposition 3.4, we may take  $\mathcal{O}_Y(S)$  to be any prescribed line bundle of degree  $n$  if  $n$  is large enough.  $\square$

**3.3. The general case.** We now use the results of § 3.1 and § 3.2 to deduce the main theorem. Denote by  $H_{d,g}(Y)$  the moduli stack of degree  $d$  and genus  $g$  branched covers of  $Y$  and by  $\text{Vec}_{d-1,g+d-1}(Y)$  the moduli stack of vector bundles of rank  $d - 1$  and degree  $g + d - 1$  on  $Y$ . The stack  $H_{d,g}(Y)$  is Deligne–Mumford and of finite type and the stack  $\text{Vec}_{d-1,g+d-1}(Y)$  is Artin and locally of finite type. Both stacks are smooth. The rule  $\phi \mapsto E_\phi$  gives a morphism

$$\tau : H_{d,g}(Y) \rightarrow \text{Vec}_{d-1,g+d-1}(Y).$$

**Theorem 3.9.** *Let  $E$  be a vector bundle on  $Y$ . There exists  $n$  (depending on  $E$ ) such that for any line bundle  $L$  of degree at least  $n$ , there exists a smooth curve  $X$  and a finite flat morphism  $\phi : X \rightarrow Y$  such that  $E_\phi \cong E \otimes L$  and such that the map  $\tau$  is smooth at  $\phi$ .*

*Proof.* We begin by analyzing the map  $\tau$  on first order deformations. Let  $[\phi : X \rightarrow Y]$  be a point of  $H_{d,g}(Y)$  and set  $E = E_\phi$ . The space of first order deformations of  $\phi$  is given by

$$\text{Def}_\phi = H^0(X, N_\phi),$$

where  $N_\phi = \text{coker}(T_X \rightarrow \phi^* T_Y)$ . The space of first order deformations of  $E$  is given by

$$\text{Def}_E = H^1(Y, \text{End } E).$$

Consider the canonical affine embedding  $X \subset E$ . We have an exact sequence

$$0 \rightarrow T_{E/Y}|_X \rightarrow N_{X/E} \rightarrow N_\phi \rightarrow 0.$$

Note that  $T_{E/Y}|_X = \phi^* E$ . The long exact sequence on cohomology gives a map

$$H^0(X, N_\phi) \rightarrow H^1(X, \phi^* E) = H^1(Y, E \oplus \text{End } E).$$

By composing with the projection  $H^1(Y, E \oplus \text{End } E) \rightarrow H^1(Y, \text{End } E)$ , we get a map

$$H^0(X, N_\phi) \rightarrow H^1(Y, \text{End } E).$$

It is straightforward to check that this is the map on the first order deformations induced by  $\tau$ . Note in particular that if  $H^1(X, N_{X/E}) = 0$ , then  $\tau$  is surjective on first order deformations and hence smooth at  $\phi$ .

Choose an isotrivial degeneration  $E_0$  of  $E$  of the form

$$E_0 = L_1 \oplus \cdots \oplus L_{d-1},$$

where the  $L_i$ 's are line bundles with  $\deg L_i + (2g_Y - 1) \leq \deg L_{i+1}$ . Such a degeneration exists by Proposition 2.5. After replacing  $E$  by  $E \otimes \lambda$  for a line bundle  $\lambda$  of large degree, we may also assume that  $\deg L_1 \geq 2g_Y - 1$ . By Proposition 3.1, there exists a nodal curve  $X_0$  and a finite flat morphism  $\phi_0: X_0 \rightarrow Y$  with Tschirnhausen bundle  $E_0$ . By the key proposition Proposition 3.3, there exists  $n$  such that for any line bundle  $L$  of degree at least  $n$ , there exists a smooth curve  $X_1$  and a map  $\phi_1: X_1 \rightarrow Y$  with Tschirnhausen bundle  $E_1 = E_0 \otimes L$ . Furthermore, we also know that  $H^1(N_{X_1/E_1}) = 0$ . By our analysis above, we get  $\tau$  is smooth at  $\phi_1$ . Since  $E \otimes L$  isotrivially degenerates to  $E_0 \otimes L$ , we deduce that there exists  $\phi: X \rightarrow Y$  with Tschirnhausen bundle  $E \otimes L$  and such that  $\tau$  is smooth at  $\phi$ .  $\square$

Denote by  $M_{d-1, d+g-1}(Y)$  the moduli space of semistable vector bundles of rank  $(d-1)$  and degree  $(d+g-1)$  on  $Y$ . Theorem 3.9 along with the openness of smooth maps gives the following.

**Corollary 3.10.** *If  $g$  is sufficiently large (depending on  $Y$  and  $d$ ), then the Tschirnhausen bundle of a general degree  $d$  and genus  $g$  cover of  $Y$  is stable. Moreover, the rational map  $H_{d,g}(Y) \dashrightarrow M_{d-1, d+g-1}(Y)$  given by  $\phi \mapsto E_\phi$  is dominant.*

Recall the Maroni locus  $M(E) \subset H_{d,g}(Y)$  defined by

$$M(E) = \{\phi \in H_{d,g}(Y) \mid E_\phi \cong E\}.$$

**Corollary 3.11.** *Let  $E$  be a vector bundle on  $Y$  of rank  $(d-1)$  and degree  $e$ . If  $g$  is sufficiently large (depending on  $Y$  and  $E$ ), then for every line bundle  $L$  on  $Y$  of degree  $d+g-1-e$ , the Maroni locus  $M(E \otimes L)$  contains an irreducible component of the expected codimension  $h^1(\text{End } E)$ .*

*Proof.* Let  $U$  be the open subset of the Hilbert scheme of curves in  $P = \mathbf{P}(E^\vee \otimes L^\vee \oplus \mathcal{O}_Y)$  of genus  $g$ , of degree  $d$  over  $Y$ , that are smooth and disjoint from the hyperplane at infinity  $\mathbf{P}(E^\vee)$ . Every  $[X] \in U$  gives  $\phi: X \rightarrow Y$  with Tschirnhausen bundle  $E \otimes L$ . Furthermore, the map

$$U \rightarrow M(E \otimes L)$$

is surjective with fibers isomorphic to  $\text{Aut}(P/Y)$ . The normal bundle  $N_{X/P}$  is a vector bundle of rank  $(d-1)$  and degree  $(d+2)(d+g-1)$ .

By the key proposition Proposition 3.3, there exists  $[X] \in U$  with  $H^1(N_{X/P}) = 0$ . Then the dimension of  $U$  at  $[X]$  is given by

$$\dim_{[X]} U = \chi(N_{X/P}) = d^2 + 2d + 3g - 3.$$

We may assume  $\deg L$  to be large enough so that  $H^1(E \otimes L) = 0$  and  $H^0(E^\vee \otimes L^\vee) = 0$ . Then

$$\dim \text{Aut}(P/Y) = d^2 + g - 1 + h^1(\text{End } E).$$

It follows that the component of  $M(E \otimes L)$  containing  $[\phi : X \rightarrow Y]$  has dimension

$$2g + 2d - 2 - h^1(\text{End } E) = \dim H_{d,g} - h^1(\text{End } E).$$

□

#### 4. HIGHER DIMENSIONS

It is important to emphasize the difference between Question 1.1 and Theorem 1.2: the former asks for the existence of *some* twist of  $E$  to be a Tschirnhausen bundle, whereas the latter asks that all sufficiently positive twists of  $E$  are Tschirnhausen bundles. By this, we mean the following:

**Question 4.1.** *Let  $Y$  be a smooth projective variety,  $L$  an ample line bundle, and  $E$  a vector bundle of rank  $(d - 1)$ . Is  $E \otimes L^n$  a Tschirnhausen bundle for all sufficiently large  $n$ .*

However, it is simple to see that the answer to Question 4.1 is negative, at least without additional hypotheses.

**Example 4.2.** Take  $Y = \mathbf{P}^4$ , and  $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$ . Then a sufficiently positive twist  $E'$  cannot be the Tschirnhausen bundle of a smooth branched cover  $X$ : the rank 4 vector bundle  $\text{Sym}^3 E' \otimes (\det E')^\vee$  becomes very ample, forcing its fourth chern class to be nonzero. Thus a general section would vanish completely at some points, which in turn would lead to a positive dimensional fiber in the hypothetical branched cover. In fact, this analysis shows that  $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$  can be a Tschirnhausen bundle of a smooth triple cover if and only if  $b = 2a$ .

This example illustrating the failure of Theorem 1.2 can be generalized for all degrees  $\geq 3$ , provided the base  $Y$  is allowed to be very high dimensional. In fact, for each degree  $d$  there is a “threshold dimension”  $m(d)$  where Question 4.1 has a negative answer for some variety  $Y$  of dimension  $m(d)$ .

The results of Lazarsfeld in [11] show that  $m(d) \leq d + 1$  for all  $d \geq 3$ .

**Proposition 4.3.** *Let  $E$  be a vector bundle of rank  $(d - 1)$  on  $\mathbf{P}^r$ , where  $r \geq d + 1$ . Then  $E$  must contain a line bundle summand. Furthermore, for all sufficiently large  $n$ ,  $E(n)$  is not a Tschirnhausen bundle of a smooth, connected cover.*

*Proof.* The proof relies on [11, Proposition 3.1] in [11] which states that for a smooth branched cover  $\phi : X \rightarrow \mathbf{P}^r$  of degree  $d \leq r - 1$ , the pullback map

$$\phi^* : \text{Pic}(\mathbf{P}^r) \rightarrow \text{Pic } X$$

is an isomorphism.

From the arguments in the proof of [11, Lemma 3.4] we deduce an isomorphism

$$\mathcal{O}_{\mathbf{P}^r} \oplus E \simeq \mathcal{O}_{\mathbf{P}^r}(l) \oplus E^\vee(l)$$

where  $l = \frac{2\det E}{d} > 0$ . This implies  $E = \mathcal{O}_{\mathbf{P}^r}(l) \oplus E'$  for some vector bundle  $E'$ .

Applying the same reasoning with  $E$  replaced by  $E(n)$  shows that  $E$  must have line bundle summands of infinitely many degrees. □

The following example shows that  $m(3) = 4$ .

**Example 4.4.** We use the well-known structure theorem on branched covers of degree 3 [4]. If  $\phi : X \rightarrow Y$  is a degree 3 branched cover with Tschirnhausen bundle  $E$ , then  $X \subset \mathbf{P}E$  is the zero locus of a section  $s \in H^0(\mathcal{O}_{\mathbf{P}E}(3) \otimes (\det E)^\vee)$ . By pushing forward to  $Y$ , this is the same as a section of the vector bundle  $V := \text{Sym}^3 E \otimes (\det E)^\vee$ .

Conversely, starting from an arbitrary rank 2 vector bundle  $E$  on  $Y$ , if we twist by a sufficiently positive line bundle  $L$ , the vector bundle  $V$  becomes very ample. If  $\dim Y \leq 3$ , then a general section  $s \in V$  will be non-vanishing everywhere on  $Y$ , and therefore the induced divisor  $X \subset \mathbf{P}E$  will be smooth, and its projection will be a finite degree 3 cover of  $Y$ .

If  $\dim Y > 3$  (and  $Y$  projective), then for sufficiently high twists of  $E$  every section of  $V$  must vanish at some point  $y \in Y$  – the resulting divisor in  $\mathbf{P}E$  will fail to have finite projection over  $y \in Y$ .

Therefore,  $m(3) = 4$ .

The next example shows that, even if we allow  $X$  to be singular but still Gorenstein,  $m(d)$  is finite.

**Example 4.5.** First, let  $\phi : X \rightarrow Y$  be an arbitrary finite, flat, degree  $d$  branched cover. Then the sheaf  $\phi_* \mathcal{O}_X$  is a sheaf of  $\mathcal{O}_Y$ -algebras, and it splits as  $\phi_* = \mathcal{O}_Y \oplus E^\vee$ .

Suppose over some point  $y \in Y$ , the multiplication map

$$m : \text{Sym}^2 E^\vee \rightarrow \phi_* \mathcal{O}_X$$

is identically zero. Then, writing  $k = k(y)$  as the residue field, it follows that

$$(\phi_* \mathcal{O}_X) \otimes k(y) \simeq k[x_1, \dots, x_{d-1}] / (x_1, \dots, x_{d-1})^2,$$

i.e.  $\phi^{-1}(y)$  is isomorphic to the length  $d$  fat point, defined by the square of the maximal ideal of the origin in an affine space. When  $d \geq 3$ , these fat points are not Gorenstein. Since  $Y$  is smooth, this implies  $X$  is not Gorenstein.

Now, if  $E$  is a vector bundle on  $Y$  and  $L$  is a sufficiently positive line bundle, then the bundle

$$M := \text{Hom}(\text{Sym}^2(E \otimes L)^\vee, \mathcal{O}_Y \oplus (E \otimes L)^\vee)$$

becomes very ample, and therefore a general section  $m \in H^0(Y, M)$  will vanish identically at some points  $y \in Y$ , provided

$$\dim Y \geq \text{rk } M = d \binom{d}{2}.$$

The conclusion is: Once  $\dim Y \geq d \binom{d}{2}$ , Question 4.1 has a negative answer even if we relax the smoothness assumption on  $X$  and allow arbitrary Gorenstein schemes.

**4.1. Modifications of the original question.** Following the discussion in the previous section, natural modified versions of Question 4.1 emerge.

The first obvious question is

**Question 4.6.** *Is the analogue of Theorem 1.2 true for all  $Y$  with  $\dim Y \leq d$ ?*

We can relax the finiteness assumption on  $\phi$ :

**Question 4.7.** *Let  $Y$  be a smooth projective variety,  $E$  a vector bundle in  $Y$ . Does there exist a line bundle  $L$  on  $Y$ , a smooth variety  $X$ , and a generically finite map  $\phi : X \rightarrow Y$  such that  $E \otimes L \simeq (\phi_* \mathcal{O}_X / \mathcal{O}_Y)^\vee$ ?*

*Remark 4.8.* A similar question is addressed in work of Hirschowitz and Narasimhan [7], where it is shown that any vector bundle on  $Y$  is the direct image of some line bundle on a smooth variety  $X$  under a generically finite morphism.

We can keep the finiteness requirement on  $\phi$  in exchange for smoothness of  $X$ . We end the paper with the following open-ended question:

**Question 4.9.** *What singularity assumptions on  $X$  (or the fibers of  $\phi$ ) yield a positive answer to Question 4.1?*

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